

# Solution of Sondow's problem: a synthetic proof of the tangency property of the parbelos

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October 23, 2012

## Abstract

In a recent paper titled *The parbelos, a parabolic analog of the arbelos*, Sondow asks for a synthetic proof to the tangency property of the parbelos. In this paper, we resolve this question by introducing a converse to Lambert's Theorem on the parabola and in the process prove some new properties of the parbelos.

## 1 Introduction

In a recent paper, Jonathan Sondow introduced the parbelos - a parabolic analogue of the arbelos [1]. One of the beautiful properties of the parbelos is that the tangents at the cusps of the parbelos form a rectangle, and that the diagonal of the rectangle opposite the cusp is tangent to the upper parabola. Moreover, the tangency point lies on the bisector of the angle at the cusp. Sondow asks for a synthetic proof of these two properties of the tangent rectangle of the parbelos, which he proves by analytic means. In this paper, we present such a proof. We do so by introducing a converse to the following Theorem of Lambert: the circumcircle of a triangle formed by three tangent lines to the parabola passes through the focus of the parabola. In the process of proving Sondow's tangency property, we discover some new properties of the parbelos.

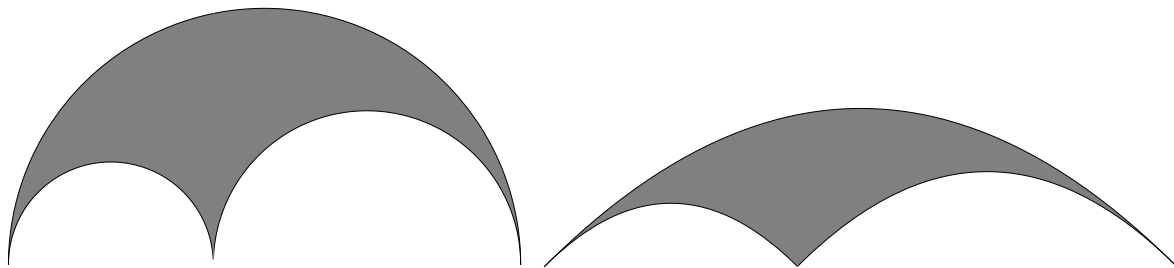


Figure 1.1: The arbelos and the parbelos.

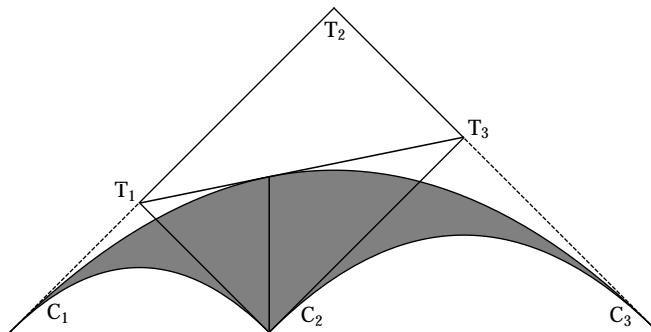


Figure 1.2: Sondow's Tangency Property: the diagonal  $T_1T_3$  of the tangent rectangle  $C_2T_1T_2T_3$  is tangent to the outer parabola. Moreover, the tangency point is the intersection of the angle bisector of cusp  $C_2$  with the outer parabola.

## 2 Preliminaries

The classical Simson-Wallace Theorem is a useful tool in understanding the parabola. It states that

**Theorem 1.** (*Simson-Wallace Theorem*) *Given a triangle  $\triangle ABC$  and a point  $P$  in the plane, the orthogonal projections of  $P$  into the sides (also called pedal points) of the triangle are collinear if and only if  $P$  is on the circumcircle of  $\triangle ABC$  [2].*

In general, a pedal curve is defined as the locus of orthogonal projections of a point into the tangents of the curve. In a sense discussed in [3], the parabola may be viewed as a polygon with infinitely many vertices which satisfies the following Simson-type property: it is the unique curve such that its pedal curve with respect to a point is a line. The point turns out to be the focus  $F$  of the parabola and the line is the supporting line at its vertex, which we will denote by  $\Lambda$ .

**Theorem 2.** *A line  $l$  is tangent to the parabola if and only if the orthogonal projection of the focus  $F$  into  $l$  lies on the supporting line  $\Lambda$ .*

*Proof.* For a proof of the “only if” statement, we refer the reader to [3] and [4]. Let  $P$  be the orthogonal projection of  $F$  into  $l$  and assume that  $P \in \Lambda$ . If  $P$  is the vertex of  $G$ , then clearly  $l = \Lambda$  and we are done. So assume otherwise. Since  $\Lambda$  has no points inside of the parabola, there exists a tangent line  $\tilde{l}$  to  $G$  not equal to  $\Lambda$  which passes through  $P$ . The “only if” part implies that the orthogonal projection of  $F$  into  $\tilde{l}$  is on  $\Lambda$ , and is therefore  $P$ . It follows that  $l = \tilde{l}$ .

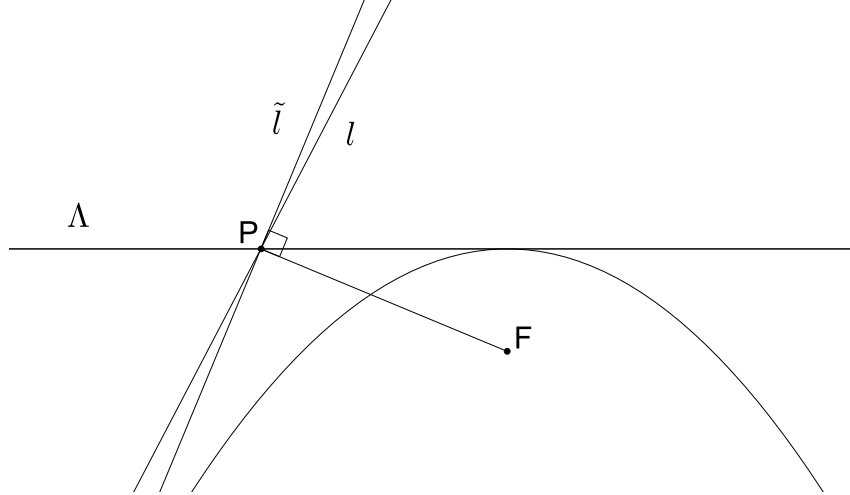


Figure 2.1: Proof of Theorem 2.

□

Lambert’s Theorem on the parabola states that the circumcircle of a triangle formed by three tangents to the parabola always passes through the focus. Using Theorem 2, we can prove the statement quite easily: let three tangents  $l_1, l_2, l_3$  to the parabola be given. Then the orthogonal projections of  $F$  into  $l_1, l_2, l_3$  all lie on  $\Lambda$ , and are therefore collinear. By the Simson-Wallace Theorem,  $F$  lies on the circumcircle of the triangle formed from  $l_1, l_2, l_3$ . We introduce a converse to Lambert’s Theorem:

**Theorem 3.** (*Converse to Lambert’s Theorem*) *Let  $l_1$  and  $l_2$  be two distinct lines tangent to a parabola  $G$  with focus  $F$ . Let  $I = l_1 \cap l_2$  be their intersection and consider any circle  $C$  passing through points  $F$  and  $I$ . Let  $H_i \in C \cap l_i$ , for  $i = 1, 2$  with at least one  $H_i \neq I$ . Then line  $H_1H_2$  is tangent to  $G$ .*

*Proof.* Observe that if  $C \cap l_i = \{I\}$  for  $i = 1, 2$ , then  $C$  must be tangent to both  $l_1$  and  $l_2$  at  $I$ , so that  $l_1 = l_2$ . Thus we may always pick at least one  $H_i$  to be not  $I$ . If  $H_i = I$  for some  $i$ , then the statement clearly holds. So assume that  $H_i \neq I$  for each  $i$ . By Theorem 2, the orthogonal projections of  $F$  into  $l_1$  and  $l_2$  lie on  $\Lambda$ . Since  $F$  is on the circumcircle of  $\triangle H_1H_2I$ , its pedal is a line (by Theorem 1). As a line is uniquely determined by two points, this line must be  $\Lambda$ . Applying Theorem 2 again yields that  $H_1H_2$  is tangent to  $G$ .

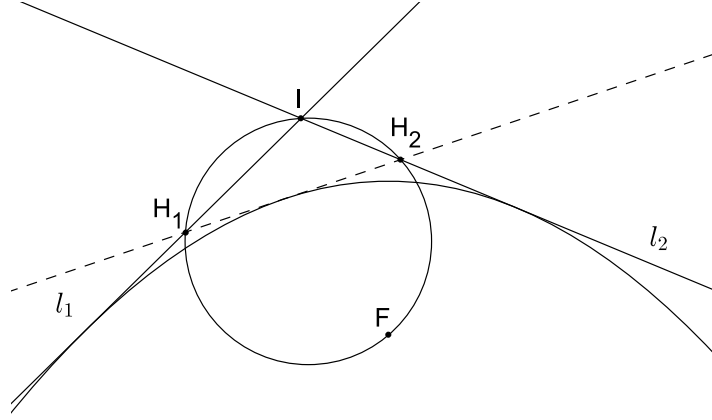


Figure 2.2: Theorem 3.

□

### 3 Parbelos

Recall that the latus rectum of a conic is the chord through the focus parallel to the conic's directrix. The parbelos is constructed as follows: given three points  $C_1, C_2, C_3$  on a line, construct parabolas  $G_1, G_2, G_3$  that open in the same direction and whose latera recta are  $C_1C_2$ ,  $C_2C_3$  and  $C_1C_3$ , respectively.

The tangent line of a parabola at either endpoint of its latus rectum forms an angle of  $\frac{\pi}{4}$  with the latus rectum. As such, parabolas  $G_1$  and  $G_2$  share the same tangent at  $C_1$ , and similarly parabolas  $G_2$  and  $G_3$  share a tangent at  $C_3$ . At cusp  $C_2$ , however, we obtain two different tangent directions. One can extend these four tangents to form a rectangle whose vertices are the intersections of tangent lines as in Figure 1.2. We will denote the vertices of this rectangle by  $C_2, T_1, T_2, T_3$ .

In his paper [1], Sondow asks for a synthetic proof of the following Theorem, which he proves via analytic geometry:

**Theorem 4.** (*Sondow's Tangency Property*) *In the tangent rectangle of the parbelos, the diagonal opposite the cusp is tangent to the upper parabola. The contact point lies on the bisector of the angle at the cusp.*

*Proof.* Let us inscribe the tangent rectangle  $r = T_1T_2T_3C_2$  in another rectangle  $R$  whose sides are parallel and orthogonal to  $C_1C_3$ . At the cusp  $C_2$ , the angles formed between  $C_1C_3$  and  $C_2T_1$ ,  $C_2T_3$  are  $\frac{\pi}{4}$  and  $\frac{3\pi}{4}$ . Using right triangles, it is easy to see that  $R$  must be a square and its center  $O$  is the same as that of  $r$ .

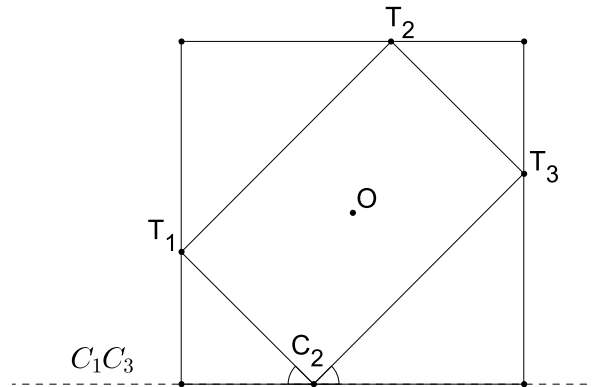


Figure 3.1: Rectangle  $R$  circumscribing the tangent rectangle  $r$ .

Consider the circumscribing circle of  $r$ .

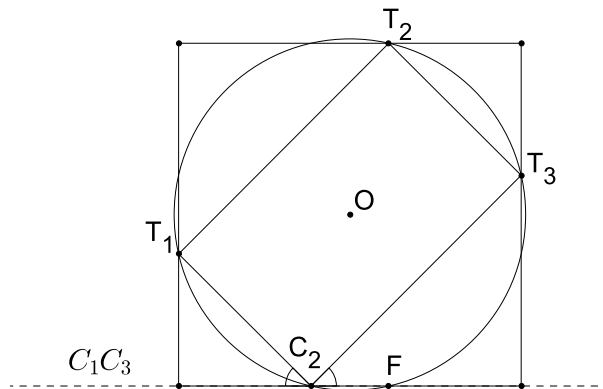


Figure 3.2: The angles at the cusp  $C_2$  are equal.

Since its center is  $O$ , by symmetry it intersects  $C_1C_3$  at a point  $F$ , such that  $F$  is the orthogonal projection of  $T_2$  down to  $C_1C_3$ . This point is the focus of the outer parabola. Since  $F, C_2, T_1, T_2, T_3$  lie on a circle, Theorem 3 implies that  $T_1T_3$  is tangent to the parabola.

As for the angle bisector at the cusp  $C_2$ , the billiard angle property implies that it is orthogonal to  $C_1C_3$  (see Figure 3.3). Let  $H$  be the intersection of the angle bisector with the top of the rectangle (i.e., with the directrix of the outer parabola), as in Figure 3.4. We would like to see that  $FT = HT$ .

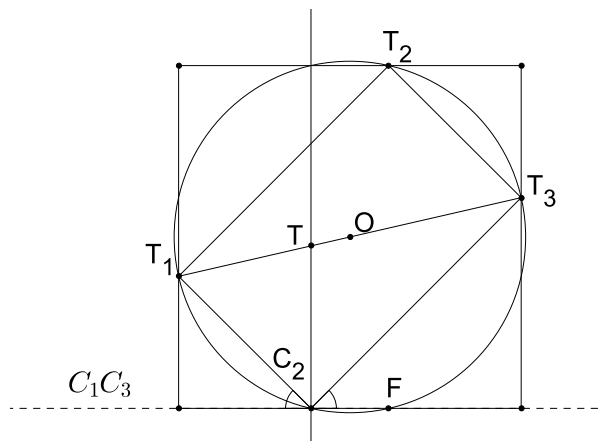


Figure 3.3: The angle bisector at cusp  $C_2$  is orthogonal to  $C_1C_3$ .

This is not too hard to see from the diagram below.

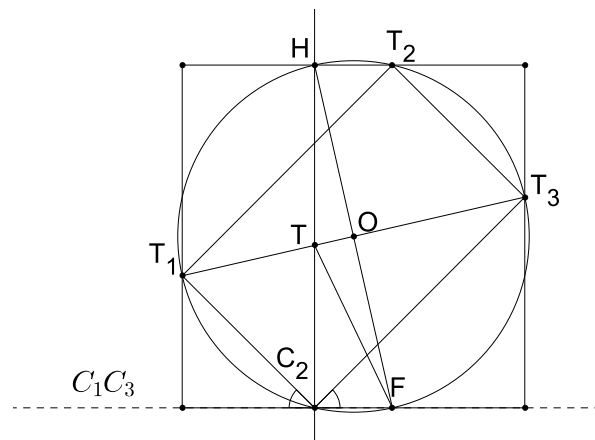


Figure 3.4:  $FT = HT$ .

It also follows from the diagram that  $F$  is equidistant from  $T_1$  and  $T_3$ . □

*Remark 5.* One way to look at the configuration in Figure 3.1 is as a 4-periodic billiard trajectory in a square billiard table (for a survey on mathematical billiards, see [5]). It would be interesting to see whether there is a deeper connection between (p)arbelos and billiards.

Let  $A_1$  be the intersection of the axis of symmetry of  $G_1$  with the directrix of  $G_3$  (i.e., the line parallel to  $C_1C_3$  and passing through  $T_3$ ). Define  $A_3$  similarly. From the proof, it is easy to see that the following previously undescribed facts hold for the parbelos:

**Corollary 6.** 1. The focus  $F$  of the outer parabola is equidistant from vertices  $T_1$  and  $T_3$  of the tangent rectangle.

2. The intersection  $H$  of the angle bisector at cusp  $C_2$  and the directrix of the outer parabola lies on the circumcircle  $F, C_2, T_1, T_2, T_3$  of the tangent rectangle.

3. This point  $H$  is equidistant from vertices  $T_1$  and  $T_3$ .

4. Points  $A_1$  and  $A_3$  lie on circle  $F, C_2, T_1, T_2, T_3$ .

5. Point  $A_1$  is equidistant from  $C_2$  and  $T_2$  and so is point  $A_3$ .

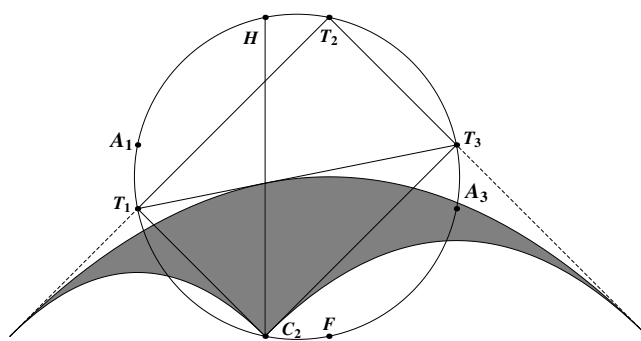


Figure 3.5: The circumcircle of the tangent rectangle and notable points lying upon it.

## References

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- [2] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*. MAA, New York, 1967.
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- [4] Hilbert, D. and Cohn-Vossen, S. *Geometry and the Imagination*. New York: Chelsea, 1999. Pages 26-27.
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